NOTES

Hex Must Have a Winner: An Inductive Proof

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The game of Hex is an excellent example of a game for which a winning strategy is known to exist, even though it is not known what the strategy is. It is easy to show the existence of a strategy once it is known that either black or white must win, that is, that Hex cannot end in a draw. The available proofs of the latter fact are all rather difficult (see, for instance, [**1**, pp. 334–338]). In this note we give a simple proof.

A Hex board is a parallelogram divided into m rows and n columns of hexagons. Players alternate turns placing black and white stones on the board with the objective of completing a chain from one side of the board to the opposite side, one player seeking a chain joining top to bottom, the other joining right to left. We will show that the game cannot end without a winner. Specifically, whenever a Hex board is completely filled with black and white stones, there must be either a black chain from right to left or a white chain from top to bottom. And, equivalently, there must be either a white chain from right to left or a black chain from top to bottom. Consequently, one of the players must have achieved his objective and won the game.

We will prove our proposition by induction on m and n, the dimensions of the board. The proposition is clear for a $1 \times n$, $m \times 1$, or 2×2 board. Now we assume it true for any board smaller than $m \times n$. Consider the $(m-1) \times n$ board obtained by deleting row m. By the inductive hypothesis there is either a black chain from column 1 to column n or else a white chain W_1 from row 1 to row $m-1$. In the former case we are done, so we assume the latter. It follows from an analogous argument involving deletion of row 1 that there must be a white chain W_2 from row 2 to row m. We assume that W_1 and W_2 do not meet, or else we would be done.

By deleting column n and then column 1 we can show in a similar manner that there are nonintersecting black chains B_1 from column 1 to column $n-1$ and B_2 from column 2 to column n. Since these horizontal black chains do not meet, the number of rows m must be greater than 2.

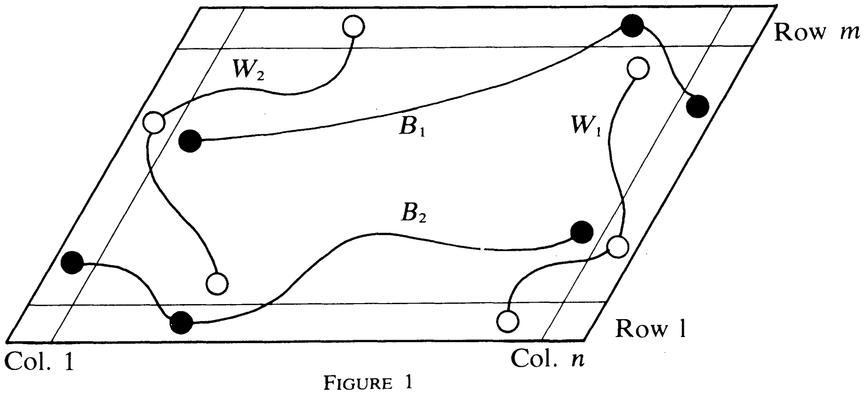

FIGURE 1

Similarly, since the vertical chains W_1 and W_2 do not meet, the number of columns n must be greater than 2.

We now consider the $(m-2)\times(n-2)$ board (See FIGURE 1) obtained by deleting rows 1 and m and columns 1 and n. In this board we apply the inductive hypothesis in its equivalent form: there must be either a white chain from column 2 to column $n-1$ or a black chain from row 2 to row $m-1$. We assume, without loss of generality, the former. This chain W_3 must intersect chains W_1 and W_2, so W_1, W_2 and W_3 together form a white chain from row 1 to row m. This completes the proof.

We note, in conclusion, that this proof can easily be modified to deal with other games of this sort, for instance, Bridge-it.

Reference

[1] Anatole Beck, Michael Bleicher, and Donald Crowe, Excursions into Mathematics, Worth, New York, 1969.

A Double Butterfly Theorem

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To the extensive annals of geometric lepidopterology we add a further modification of the well-known butterfly problem. Let us define a "**butterfly**," denoted by $)B($, as the two triangles formed by the diagonals and two opposite sides of a convex quadrilateral, and refer to these triangles as

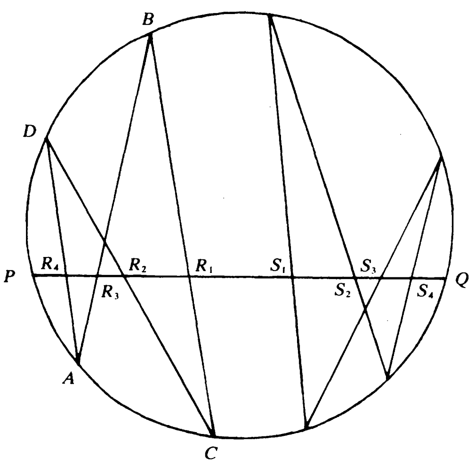

FIGURE 1.

"wings" of the butterfly. FIGURE 1, depicting two butterflies inscribed in a circle, illustrates our main result:

THEOREM. *Let PQ be a fixed chord of a circle. Let $)R($ and $)S($ be inscribed in the circle and oriented such that their wings cut PQ (in order from left to right) at R_4, R_3, R_2, R_1, and S_1, S_2, S_3, S_4, respectively. If $PR_1=QS_1$, $PR_2=QS_2$, and $PR_3=QS_3$, then $PR_4=QS_4$.*

Proof. Consider $)R($. Denoting by $(UVWX)$ the double ratio on points U, V, W, and X, we have

$$(PR_4R_3Q)=\frac{\sin\angle PAB}{\sin\angle BAD}\div\frac{\sin\angle PAQ}{\sin\angle QAD},\qquad (PR_2R_1Q)=\frac{\sin\angle PCB}{\sin\angle BCD}\div\frac{\sin\angle PAQ}{\sin\angle QCD}.$$