An Inductive Proof of Hex Uniqueness

Samuel Clowes Huneke

Abstract. A short, inductive proof is presented of the fact that a Hex board cannot be colored such that winning conditions are satisfied for both players.

It is well known that the game of Hex, independently invented by Piet Hein and John Nash, always has a winner. In it, two players, Black and White, attempt to connect opposite sides—East and West or North and South—of a parallelogram tiled with hexagons by coloring tiles with their respective colors (see Figure 1). We call these sides for the respective players *necessary edges*. Similar to this result is the intuitively obvious fact that the board cannot be colored such that there are two winners. In 1979, David Gale mentioned a proof of this by "induction on the size of the board," but did not present it [**4**, p. 820]. In fact, no such inductive proof has, to our knowledge, ever been published.

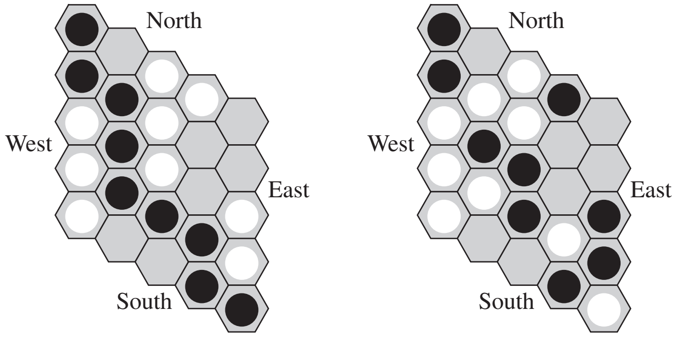

Figure 1. Two examples of 5 × 5 Hex boards. Black has won on the left, and White on the right.

We therefore present our own proof of Hex Uniqueness, inspired by David Berman's inductive proof of the fact that Hex always has a winner (the "Hex Theorem") [**2**].

Theorem 1 (Hex Uniqueness Theorem). *It is impossible for any Hex Board to be colored in such a way as to satisfy winning conditions for more than one player.*

Proof. It is easily demonstrable that Hex Uniqueness holds for any $2 \times m$, $n \times 2$, and smaller-dimensional boards.

We therefore assume it true for all $i \times j$ boards, with $i < n$ and $j < m$, $i = n$ and $j < m$, or $i < n$ and $j = m$. Further, imagine an $n \times m$ board ($H(n, m)$) colored such that both Black and White have won. Each player therefore has a winning path connecting opposite sides of the board. More specifically, each player has a *minimal path*, which we define as a winning path **M** contained in a given winning path such that **M** contains precisely one hexagon adjacent to each necessary edge; these hexagons in

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turn are each adjacent to precisely one other hexagon on **M**, and all other hexagons contained in **M** border precisely two other component hexagons. We leave it to the reader to certify that such a minimal path is indeed contained in any winning path.

First, consider Black. Because he has a path from East to West, we can remove the nth column from the board and Black will retain a winning path. However, by our above assumption, there can be only one winner on this new $n - 1 \times m$ board. Hence, White's minimal path on $H(n, m)$ must contain a hexagon in the nth column. We follow the same argument to show that White's minimal path must contain a hexagon in the first column. Hence, on the $n - 2 \times m$ board created by removing the first and last columns, White retains a path **P** connecting East and West. Note that none of the hexagons contained in **P** may be in the first or final rows; were one to be contained therein, then that hexagon, bordering a necessary edge, would be adjacent to two other hexagons on the minimal path, contradicting our definition above.

Consider White's position and remove the first row of $H(n, m)$. Because White retains a winning path on the new $n \times m - 1$ board, Black cannot win on it, meaning that Black's original minimal path must contain a hexagon in the first row. Similarly, Black's minimal path must contain a hexagon in the mth row. Thus, by the same argument as above, on the $n \times m - 2$ board created by removing the first and last rows, Black has a path connecting North and South, no hexagon of which can be contained in the first or final columns.

We now remove the first column, the first row, the nth column, and the mth row to create an $n - 2 \times m - 2$ board. Note that White has a path connecting East and West and Black a path connecting North and South. Imagine that all black tiles are white and all white tiles black. Then, winning conditions would be satisfied for both players on this $n - 2 \times m - 2$ Hex board, contradicting our assumption for smaller-dimensional boards.

Hence, no coloring exists for any Hex board that satisfies winning conditions for more than one player. ■

Additional proofs of Hex Uniqueness, many of which remain unpublished, rely on the non-planarity of K_5, the Jordan Curve Theorem, or the Four Color Theorem [**4, 5, 7**]. Moreover, both John Pierce and Anatole Beck et al. presented proofs of Hex Uniqueness in 1961 and 1969, respectively, as part of their broader proofs of the Hex Theorem [**1, 8**]; however, neither successfully proves Hex Uniqueness.[1] On the other hand, Ryan B. Hayward and Jack van Rijswijck prove Hex Uniqueness via the game of Y [**6**, p. 2518]. Their paper, however, only provides the sketch of a proof, the details of which can be found in [**3**]. We believe that our proof, using a different approach, has the benefit of simplicity.

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[1]Beck et al. prove only 'winning conditions not satisfied for Black' ⇒ 'winning conditions satisfied for White' (i.e., the Hex Theorem), while Pierce's argument consists of the claim "It is clearly impossible for white to have won also, for the continuous band of adjacent black cells from the left border to the right precludes a continuous band of white cells to the bottom border" [**8**, pp. 12–13].

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Department of Mathematics, The London School of Economics and Political Science, London, England
huneke@alumni.lse.ac.uk

Direct Proof of the Uncountability of the Transcendental Numbers

We usually prove that the set of the real transcendental numbers $\mathbb{R} \setminus \mathbb{A}$ is uncountable *indirectly* by proving that the set of the algebraic numbers $\mathbb{A}$ is countable. Here, we present a *direct* proof that $\mathbb{R} \setminus \mathbb{A}$ is uncountable.

Theorem. *The set* $\mathbb{R} \setminus \mathbb{A}$ *is uncountable.*

Proof. The function $f : [0, +\infty) \to \mathbb{R} \setminus \mathbb{A}$ defined by

$$f(x) = \begin{cases} \pi + x \text{ if } \pi + x \notin \mathbb{A} \\ \pi - x \text{ if } \pi + x \in \mathbb{A} \end{cases}$$

is: (1) *well-defined*, because if $\pi + x \in \mathbb{A}$, then $\pi - x \notin \mathbb{A}$, otherwise

$$\pi = \frac{(\pi + x) + (\pi - x)}{2} \in \mathbb{A}$$

which is false; (2) *injective*, because if $f(x) = f(y)$, then $x = |f(x) - \pi| = |f(y) - \pi| = y$. (The function f is inspired by the folklore theorem "$\pi + e \notin \mathbb{A}$ or $\pi - e \notin \mathbb{A}$", and its proof "otherwise $\pi = \frac{(\pi+e)+(\pi-e)}{2} \in \mathbb{A}$ which is false".) ■

—Submitted by Jaime Gaspar

At the time of writing: INRIA Paris-Rocquencourt, πr^2, Univ Paris Diderot, Sorbonne Paris Cité, F-78153 Le Chesnay, France. Financially supported by the French Fondation Sciences Mathématiques de Paris. At the time of publication: Universitat Rovirai Virgili, Department of Computer Engineering and Mathematics, Av. Països Catalans 26, E-43007 Tarragona, Catalonia. jaime.gaspar@urv.cat. Centro de Matemática e Aplicações (CMA), FCT, UNL.
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